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Estimating the Change Point of a Poisson Process with the Gravitational Search Algorithm: Maximum Likelihood and Schwarz Information Criterion Approaches

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ABSTRACT

When a control chart signals that a process is out of control, identifying the time at which the process actually changed (the change point) focuses the search for the assignable cause. This paper considers a Poisson process monitored by a Shewhart C chart and subject to a single step change in its rate. Two change-point estimators are studied, the maximum likelihood estimator (MLE) and an estimator based on the Schwarz information criterion (SIC), and we examine the gravitational search algorithm (GSA) for estimating the change point and the rate parameters jointly when the out-of-control rate is restricted to a search interval. In a reproducible Monte Carlo study ($\lambda_0 = 20$, $\tau = 100$, $\delta = 0.5$–2.0, 10,000 replications per scenario), the conventional estimators were compared with an exact constrained estimator, obtained by enumerating all candidate change points, under three interval specifications, and with the GSA solution. An interval of ±10% around the true out-of-control rate markedly reduced the bias for small shifts ($\delta = 1.1$: MLE 103.2 vs. 108.7; SIC 105.8 vs. 120.5) but raised the probability of locating the change within ±1 subgroup by only 0.010–0.034. Wider intervals reduced these gains, and with only the direction of the shift known the constrained and conventional estimators were practically indistinguishable. GSA reached the exact optimum in 35–93% of replications, and its estimates did not differ from those of the faster, deterministic enumeration. The gains of metaheuristic change-point estimators therefore come from the information in the search interval, not from the search algorithm.

1. INTRODUCTION

Control charts are the principal tool of statistical process control (SPC) for distinguishing common-cause from assignable-cause variation [1]. When a chart signals an out-of-control condition, quality engineers begin to search for the assignable cause or causes and to remove them. This search is effective only if the time at which the process actually changed, the change point, is correctly identified. The chart's signal usually occurs some time after the

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change, especially when the shift is small. A search that begins at the signal time may therefore examine the wrong production period and miss the true cause [2].

In this paper, the change in a Poisson process is assumed to be a single step change: at an unknown time the rate parameter shifts from its in-control value $\lambda = \lambda_0$ to an out-of-control value $\lambda = \lambda_a$ ($\lambda_a \neq \lambda_0$). The process remains in that state until the assignable cause is identified and removed. Poisson processes describe the number of nonconformities per inspection unit and are monitored with the Shewhart C chart [1].

Samuel and Pignatiello [3] proposed a likelihood-based change-point estimator for a Poisson random variable subject to a step change in the rate parameter and evaluated it together with the C chart. Their estimator was more accurate and more precise than the change-point estimates implicit in CUSUM [4] and EWMA [5] charts. The same approach had been developed for the mean of a normal process [6]. Perry et al. [7] derived the MLE for a Poisson rate that changes according to a linear trend and showed that, under such disturbances, it outperforms the step-change estimator of Samuel and Pignatiello [3]. Amiri and Allahyari [2] review the broader literature on change-point estimation for post-signal diagnostics.

Information criteria offer an alternative route to change-point estimation. The Akaike information criterion (AIC) was proposed by Akaike [8] for selecting among statistical models; it is based on the Kullback–Leibler information between the fitted and the true model. Schwarz [9] proposed a modification with a stronger penalty for model complexity, the Schwarz (Bayesian) information criterion (SIC). Chen and Gupta [10] developed SIC-based procedures for detecting and locating change points in a range of parametric models, including the Poisson model.

Both the MLE and the SIC estimator require maximizing (or minimizing) an objective function over the change point and the unknown parameters. For a single change in a single parameter, this can be done by enumerating all candidate change points. Enumeration becomes impractical, however, as the number of parameters, change points or model components grows. Population-based metaheuristics provide a general-purpose alternative. The gravitational search algorithm (GSA) is one such method; it is inspired by Newton's laws of gravitation and motion and has proved competitive with other metaheuristics on a wide range of optimization problems [11].

The objective of this paper is to apply GSA to the estimation of a step change point in a Poisson process, using both the likelihood function and the SIC as objective functions, and to compare the resulting estimators by simulation not only with the conventional MLE and SIC estimators but also with the exact constrained estimators that solve the same optimization problem by enumeration. This comparison separates the contribution of the search algorithm from the contribution of the search interval imposed on the rate parameters. The contributions are: (i) a GSA-based formulation that estimates the change point and the unknown rate parameters jointly and does not limit the number of parameters; (ii) a Monte Carlo comparison of accuracy and precision for increases and decreases of the rate under both criteria; and (iii) an analysis, based on exact constrained benchmarks and three interval specifications, of the source of the observed gains and of their practical implications.

The remainder of the paper is organized as follows. Section 2 presents the MLE and SIC change-point estimators. Section 3 describes GSA and its application to change-point estimation. Section 4 describes the simulation design, and Section 5 reports the results. Section 6 discusses the findings, and Section 7 concludes the paper.

2. CHANGE-POINT ESTIMATORS

2.1. Model

Consider a Poisson process with rate λ that is initially in control ($\lambda = \lambda_0$) and, at an unknown time, changes to an out-of-control state ($\lambda = \lambda_a$). Once the step change has occurred, the process remains in the new state until the cause is identified and removed. For subgroups $i = 1, 2, \ldots, \tau$, the Poisson parameter equals the known value λ_0. For subgroups $i = \tau + 1, \tau + 2, \ldots, T$, it equals the unknown value λ_a. Here T is the subgroup at which the C chart first signals, and the unknown change point τ is the last subgroup drawn from the in-control process. Let x_i denote the number of nonconformities observed in subgroup i.

2.2. Maximum likelihood estimator

Given that the change occurred at τ, the likelihood function is

$$L(\tau, \lambda_a) = \prod_{i=1}^{\tau} \frac{e^{-\lambda_0}\lambda_0^{x_i}}{x_i!} \prod_{i=\tau+1}^{T} \frac{e^{-\lambda_a}\lambda_a^{x_i}}{x_i!}. \tag{1}$$

The log-likelihood is

$$\ln L(\tau, \lambda_a) = -\sum_{i=1}^{T} \ln x_i! - \tau\lambda_0 + \ln\lambda_0 \sum_{i=1}^{\tau} x_i - (T-\tau)\lambda_a + \ln\lambda_a \sum_{i=\tau+1}^{T} x_i. \tag{2}$$

For a given τ, the value of λ_a that maximizes (2) is the mean of the post-change observations:

$$\hat{\lambda}_a(\tau) = \frac{1}{T-\tau}\sum_{i=\tau+1}^{T} x_i. \tag{3}$$

Substituting (3) into (2) and omitting terms that do not depend on τ gives the MLE of the change point [3]:

$$\hat{\tau}_{MLE} = \underset{0\le\tau\le T-1}{\operatorname{argmax}} \left[-\tau\lambda_0 - S_\tau + S_\tau \ln\left(\frac{S_\tau}{(T-\tau)\lambda_0}\right)\right], \qquad S_\tau = \sum_{i=\tau+1}^{T} x_i. \tag{4}$$

2.3. Schwarz information criterion estimator

For a model with *p* free parameters fitted to *n* observations, AIC and SIC are defined as

$$AIC = -2\ln L(\hat{\theta}) + 2p, \qquad SIC = -2\ln L(\hat{\theta}) + p\ln n. \tag{5}$$

The model with the smallest criterion value is preferred. Following Chen and Gupta [10], the observations are assumed to satisfy $X_1, \ldots, X_\tau \sim$ Poisson(λ_0) and $X_{\tau+1}, \ldots, X_T \sim$ Poisson(λ_a), where T is the subgroup at which the C chart signals. In this formulation λ_0 is also treated as unknown. The hypotheses are

$$H_0: \lambda_1 = \lambda_2 = \cdots = \lambda_T = \lambda_0 \quad \text{versus} \quad H_1: \lambda_1 = \cdots = \lambda_\tau = \lambda_0 \neq \lambda_{\tau+1} = \cdots = \lambda_T = \lambda_a, \tag{6}$$

where τ is the unknown location of the change. Under H_0 there is one parameter, estimated by $\hat{\lambda}_0 = M/T$ with $M = \sum_{i=1}^{T} x_i$, so

$$SIC(T) = -2\ln L_0(\hat{\lambda}_0) + \ln T. \tag{7}$$

Under H_1, with $M_\tau = \sum_{i=1}^{\tau} x_i$, $M'_\tau = \sum_{i=\tau+1}^{T} x_i$, $\hat{\lambda}_0 = M_\tau/\tau$ and $\hat{\lambda}_a = M'_\tau/(T-\tau)$,

$$SIC(\tau) = 2M_\tau - 2M_\tau \ln\frac{M_\tau}{\tau} + 2M'_\tau - 2M'_\tau \ln\frac{M'_\tau}{T-\tau} + 2\sum_{i=1}^{T} \ln x_i! + 2\ln T. \tag{8}$$

H_0 is rejected if

$$SIC(T) > \min_{1\le\tau\le T-1} SIC(\tau), \tag{9}$$

and, because SIC(τ) decreases as the likelihood increases, the change point is estimated by

$$\hat{\tau}_{SIC} = \underset{1\le\tau\le T-1}{\operatorname{argmin}} SIC(\tau). \tag{10}$$

3. THE GRAVITATIONAL SEARCH ALGORITHM AND ITS APPLICATION

3.1. The algorithm

GSA is a memoryless, population-based optimization algorithm modeled on the motion of masses in an artificial system that evolves in discrete time [11]. Each candidate solution is an *agent* whose position in the search space represents a solution of the problem. Its mass is determined by the value of the objective function: better solutions are heavier. According to the law of gravity, each mass perceives the location and state of all other masses through gravitational attraction (Figure 1), and this force serves as the means of information exchange. Heavier agents attract the others and move slowly, which leads the population toward good regions of the search space. GSA can be applied to any optimization problem in which each solution can be represented as a position in space and the similarity between solutions can be expressed as a distance.

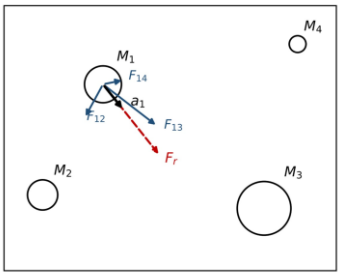


Fig. 1. Each mass is accelerated by the resultant of the gravitational forces exerted by the other masses (drawn after the description in [11]).

Consider N agents with positions $X_i = (x_i^1, \ldots, x_i^D)$ in a D-dimensional search space. At iteration t, the mass of agent i is computed from its fitness, $fit_i(t)$, as

$$m_i(t) = \frac{fit_i(t) - worst(t)}{best(t) - worst(t)}, \qquad M_i(t) = \frac{m_i(t)}{\sum_{j=1}^{N} m_j(t)}, \tag{11}$$

where $best(t)$ and $worst(t)$ are the best and worst fitness values in the population. The force exerted on agent i by agent j in dimension d is

$$F_{ij}^d(t) = G(t)\, \frac{M_i(t) M_j(t)}{R_{ij}(t) + \varepsilon} \left(x_j^d(t) - x_i^d(t) \right), \tag{12}$$

where $R_{ij}(t)$ is the Euclidean distance between the two agents, ε is a small constant, and the gravitational "constant" decreases over time as $G(t) = G_0\, e^{-\alpha t / t_{\max}}$. The total force on agent i is a randomly weighted sum of the forces exerted by the K_{best} heaviest agents, where K_{best} decreases linearly from N to 1 over the iterations:

$$F_i^d(t) = \sum_{j \in K_{best},\, j \neq i} r\, and_j\, F_{ij}^d(t). \tag{13}$$

The acceleration, velocity and position are then updated as

$$a_i^d(t) = \frac{F_i^d(t)}{M_i(t)}, \qquad v_i^d(t+1) = rand_i\, v_i^d(t) + a_i^d(t), \qquad x_i^d(t+1) = x_i^d(t) + v_i^d(t+1), \tag{14}$$

where $rand_i$ and $rand_j$ are uniform random numbers in [0, 1]. The decreasing $G(t)$ and K_{best} shift the search from exploration toward exploitation as the iterations proceed.

3.2. Application to change-point estimation

Each agent encodes a candidate solution of the change-point problem. For the MLE, the position is (τ, λ_a), and the fitness is the log-likelihood (2). For the SIC, the position is $(\tau, \lambda_0, \lambda_a)$, and the fitness is $-\mathrm{SIC}(\tau)$ from (8). The change-point coordinate is restricted to $[1, T-1]$ and rounded to the nearest integer when the fitness is evaluated. Each rate coordinate is restricted to a search interval around the parameter. The final estimate is the position of the best agent at the last iteration. Because the formulation does not rely on closed-form solutions such as (3), it places no restriction on the number of parameters and can be extended directly to models with several unknown parameters or several change points.

GSA settings: To make the search independent of the scale of the coordinates (the change-point coordinate ranges over hundreds of subgroups, the rate coordinates over a few units), each coordinate was mapped to [0, 1] before the search and mapped back when the fitness was evaluated. Preliminary tuning on 200 replications showed that this normalization is essential: without it, the change-point coordinate barely moved and GSA found the exact optimum in only 23–42% of replications. The final settings were N = 50 agents, t_{max} = 300 iterations, $G_0 = 2$ and $\alpha = 20$, with K_{best} decreasing linearly from N to 1 and $\varepsilon = 10^{-12}$. The best agent found over all iterations is returned. For the MLE the change-point coordinate ranges over $[0, T-1]$, in agreement with (4); for the SIC it ranges over $[1, T-1]$, in agreement with (10).

Search intervals: Because the search interval for the rate parameters is the decisive ingredient of the constrained estimators (Section 6.2), three specifications are studied explicitly:

- **True λa ± 10%:** $\lambda_a \in [0.9\delta\lambda_0, 1.1\delta\lambda_0]$. This interval is centred on the true out-of-control rate and therefore represents an *oracle* that is not available in practice; it corresponds to the interval used in earlier applications of metaheuristic change-point estimators.
- **True λa ± 25%:** $\lambda_a \in [0.75\delta\lambda_0, 1.25\delta\lambda_0]$, a less precise oracle.
- **Direction only:** $\lambda_a \in [\lambda_0, 2\lambda_0]$ for increases and $[0.5\lambda_0, \lambda_0]$ for decreases, which represents the realistic situation in which the engineer knows only the direction of the shift, e.g. from the side of the control limit that was crossed.

For the SIC, λ_0 was restricted to $[0.9\lambda_0, 1.1\lambda_0] = [18, 22]$ in all constrained versions.

Exact constrained estimators: For a single change point, the constrained problem can be solved exactly. For each candidate τ, the log-likelihood (2) is concave in λ_a, so its maximum over an interval [a, b] is attained at the unconstrained estimate (3) clipped to the interval, $\tilde{\lambda}_a(\tau) = \min\{\max\{\hat{\lambda}_a(\tau), a\}, b\}$. Substituting $\tilde{\lambda}_a(\tau)$ into (2) and enumerating τ gives the exact constrained MLE; the same argument applied to both rates gives the exact constrained SIC estimator. These estimators find the global optimum of the same objective that GSA searches and are therefore the appropriate benchmark for judging the search algorithm itself.

4. SIMULATION DESIGN

Monte Carlo simulation was used to compare the estimators. In each replication, 100 in-control observations were generated from a Poisson distribution with $\lambda_0 = 20$. If the C chart signalled during these 100 observations (a false alarm), the replication was discarded and the in-control sequence was regenerated, so that every retained replication corresponds to a genuine signal after the change. From observation 101 onward, observations were generated from a Poisson distribution with $\lambda_a = \delta\lambda_0$ until the C chart signalled. The true change point was therefore $\tau = 100$. All estimators were applied to the same data set. The procedure was repeated 10,000 times for each $\delta \in \{0.7, 0.8, 0.9, 1.1, 1.2, 1.3\}$, covering both decreases and increases in the rate, and for $\delta \in \{0.5, 0.6, 1.4, 1.5, 2.0\}$ for the analysis of $P(|\hat{\tau} - \tau| \le 1)$ shown in Figures 2 and 3. Because GSA requires about 15,000 fitness evaluations per replication, it was applied to the first 1,000 replications of each main scenario; its Monte Carlo standard errors are correspondingly about three times larger than those of the other estimators. All computations were carried out in Python 3 (NumPy, SciPy) with fixed seeds (seed = 1000δ); the code is available as supplementary material (cpsim.py).

The three-sigma limits of the C chart for $\lambda_0 = 20$ are

$$UCL = \lambda_0 + 3\sqrt{\lambda_0} = 33.42, \qquad LCL = \lambda_0 - 3\sqrt{\lambda_0} = 6.58. \tag{15}$$

The chart therefore signals when $x_i \ge 34$ or $x_i \le 6$. Table 1 gives the resulting signal probabilities and average run lengths (ARLs), computed exactly from the Poisson distribution. The in-control ARL is 339.7, so the probability of at least one false alarm within the first 100 subgroups is $1 - (1 - 0.00294)^{100} = 0.255$, which agrees with the observed regeneration rate (Table 2). A 10% decrease ($\delta = 0.9$) has an ARL of 650.4, longer than the in-control ARL, because the chart's limits are asymmetric relative to the Poisson distribution.

Table 1. Signal probability per subgroup and ARL of the C chart ($\lambda_0 = 20$, UCL = 33.42, LCL = 6.58).

δ	P(signal)	ARL
0.5	0.130	7.7
0.6	0.046	21.8
0.7	0.014	70.3
0.8	0.004	245.9
0.9	0.002	650.4
1.0 (in control)	0.003	339.7
1.1	0.011	94.6
1.2	0.031	31.9
1.3	0.075	13.3

1.4	0.150	6.7
1.5	0.256	3.9
2.0	0.849	1.2

Note. Exact Poisson probabilities of $X \geq 34$ or $X \leq 6$ when $\lambda = \delta\lambda_0$; $\delta = 1.0$ corresponds to the in-control state.

Table 2 summarizes the length of the simulated series. For $\delta = 0.9$ the chart signals on average 655 subgroups after the change, so the estimators must locate a change at subgroup 100 in a series of about 755 observations.

Table 2. Length of the simulated series and number of false alarms regenerated (10,000 retained replications per scenario).

δ	Mean T at signal	Median T	False-alarm regenerations
1.3	113.4	110	3,437
1.2	132.1	123	3,418
1.1	193.7	164	3,375
0.9	754.8	551	3,398
0.8	345.8	269	3,362
0.7	169.8	148	3,404

Performance was assessed with three measures: the mean of the change-point estimates over the replications ($\bar{\hat{\tau}}$), its standard error (SE = SD/√R, where R is the number of replications), and the empirical probability that the estimate lies within m subgroups of the true change point, $\hat{P}(|\hat{\tau} - \tau| \leq m)$ for m = 0, 1, 2, 3. With R = 10,000, the Monte Carlo standard error of these probabilities is at most 0.005; for the GSA results (R = 1,000) it is at most 0.016.

5. RESULTS

5.1. Accuracy and precision of the mean estimate

Table 3 reports the mean estimated change point and its standard error for all estimators, together with the proportion of replications in which GSA reached the exact constrained optimum.

Table 3. Mean estimated change point (SE) by estimator ($\tau = 100$, $\lambda_0 = 20$; 10,000 replications, GSA 1,000).

Criterion	δ	Conventional	Direction only	True λa ± 25%	True λa ± 10%	GSA, true λa ± 10%[a]	GSA agreement[b]
MLE	1.3	100.01 (0.060)	100.01 (0.061)	99.92 (0.061)	99.96 (0.044)	99.72 (0.118)	0.88
	1.2	100.97 (0.097)	100.99 (0.096)	100.68 (0.096)	100.01 (0.096)	99.40 (0.284)	0.81
	1.1	108.70 (0.275)	108.52 (0.276)	106.12 (0.259)	103.21 (0.247)	101.63 (0.689)	0.78
	0.9	101.90 (0.264)	99.69 (0.255)	99.70 (0.248)	98.72 (0.247)	98.83 (0.733)	0.93
	0.8	100.10 (0.067)	99.97 (0.067)	99.90 (0.067)	99.81 (0.064)	99.62 (0.173)	0.92
	0.7	100.02 (0.032)	99.96 (0.032)	99.96 (0.031)	99.93 (0.025)	99.77 (0.080)	0.93
SIC	1.3	99.69 (0.083)	99.99 (0.065)	99.83 (0.067)	99.96 (0.045)	99.60 (0.119)	0.82
	1.2	101.10 (0.117)	101.29 (0.110)	100.72 (0.109)	99.92 (0.106)	98.67 (0.275)	0.69
	1.1	120.46 (0.507)	121.05 (0.503)	113.89 (0.429)	105.82 (0.368)	100.86 (0.913)	0.56
	0.9	198.79 (2.976)	140.90 (1.837)	127.17 (1.467)	111.33 (0.893)	135.44 (4.509)	0.35
	0.8	100.67 (0.076)	100.45 (0.074)	100.35 (0.073)	100.19 (0.070)	100.00 (0.191)	0.78
	0.7	100.15 (0.029)	100.06 (0.029)	100.05 (0.028)	99.99 (0.026)	99.90 (0.068)	0.87

Note. [a] GSA applied to the constrained problem with the true λa ± 10% interval, first 1,000 replications. [b] Proportion of these replications in which the GSA estimate equalled the exact constrained estimate.

Conventional estimators. The conventional estimators were nearly unbiased for the larger shifts (δ = 1.3, 0.8 and 0.7) and strongly biased upward for small shifts, especially for δ = 1.1 (MLE 108.7, SIC 120.5) and, for the SIC, δ = 0.9 (198.8). The SIC estimator, which also estimates λ_0, was more biased than the MLE whenever the post-change series was long. Note that the standard errors in Table 3 refer to the mean estimate (SD/√R); the standard deviation of individual estimates is √R times larger, for example 6.0 and 27.5 subgroups for the conventional MLE at δ = 1.3 and 1.1.

Constrained estimators. Restricting λ_a to the true value ± 10% reduced the bias substantially for small shifts: for δ = 1.1, the mean estimate fell from 108.7 to 103.2 (MLE) and from 120.5 to 105.8 (SIC), and for the SIC with δ = 0.9 from 198.8 to 111.3. The reduction was about half as large with the ± 25% interval. When only the direction of the shift was known, the constrained MLE was practically identical to the conventional MLE for increases, because the unconstrained estimate (3) rarely falls outside $[\lambda_0, 2\lambda_0]$; for decreases, the direction constraint reduced the bias at δ = 0.9 (101.9 → 99.7 for the MLE; 198.8 → 140.9 for the SIC).

GSA. GSA reached the exact constrained optimum in 78–93% of replications for the MLE and in 35–87% for the SIC, with the lowest agreement for the long series at δ = 0.9 and 1.1. Because the exact constrained estimator is the global optimum of the same objective, every disagreement is a search failure. The GSA means differ from the exact constrained means by amounts compatible with Monte Carlo error and with these search failures; for SIC with δ = 0.9, GSA was considerably worse (135.4 versus 111.3).

5.2. Probability of estimating the change point within ±m subgroups

Table 4 reports the empirical probability that the estimated change point lies within m subgroups of the true change point.

The constrained estimators with the true λa ± 10% interval were consistently, but only slightly, more accurate than the conventional estimators: for m = 1 the gain was 0.010–0.034, for example 0.608 versus 0.582 (MLE, δ = 1.3) and 0.414 versus 0.387 (MLE, δ = 1.2). With the ± 25% interval the gains were between a quarter and a half as large, and with the direction-only interval they were essentially zero (at most 0.006), except for the SIC with δ = 0.9 (0.020), where the direction constraint prevents some of the extreme late estimates. The GSA probabilities were within about two Monte Carlo standard errors (≈ 0.03) of those of the exact constrained estimator in every scenario except SIC with δ = 0.9, where GSA was clearly worse for m ≥ 1.

These gains are smaller than those usually reported for metaheuristic change-point estimators. For example, for the SIC with δ = 1.2 and m = 3 the probability rose from 0.574 (conventional) to 0.613 with the exact constrained estimator and to 0.634 with GSA, and for the MLE with δ = 1.3 and m = 3 from 0.788 to 0.809 and 0.828. The differences between GSA and the exact constrained estimator are within Monte Carlo error, so none of the advantage is attributable to GSA as a search method.

Table 4. Empirical probability $\hat{P}(|\hat{\tau} - \tau| \le m)$ by estimator.

Criterion	δ	m	Conventional	Direction only	True λa ± 25%	True λa ± 10%	GSA, true λa ± 10%
MLE	1.3	0	0.351	0.351	0.357	0.369	0.391
		1	0.582	0.582	0.589	0.608	0.613
		2	0.712	0.712	0.717	0.736	0.742
		3	0.788	0.789	0.793	0.809	0.828
	1.2	0	0.198	0.198	0.203	0.211	0.223
		1	0.387	0.387	0.397	0.414	0.428
		2	0.508	0.508	0.521	0.539	0.544
		3	0.597	0.597	0.610	0.629	0.632
	1.1	0	0.058	0.059	0.063	0.068	0.066
		1	0.134	0.135	0.146	0.159	0.165
		2	0.198	0.199	0.214	0.234	0.246
		3	0.253	0.253	0.274	0.297	0.307

	0.9	0	0.074	0.076	0.076	0.078	0.068
		1	0.169	0.174	0.175	0.179	0.183
		2	0.237	0.246	0.247	0.252	0.251
		3	0.301	0.311	0.313	0.318	0.315
	0.8	0	0.247	0.249	0.250	0.253	0.253
		1	0.456	0.460	0.463	0.468	0.479
		2	0.587	0.591	0.595	0.599	0.598
		3	0.672	0.677	0.681	0.686	0.704
	0.7	0	0.444	0.449	0.450	0.458	0.455
		1	0.685	0.691	0.693	0.702	0.712
		2	0.812	0.818	0.819	0.828	0.831
		3	0.882	0.886	0.887	0.895	0.890
SIC	1.3	0	0.347	0.349	0.353	0.369	0.400
		1	0.575	0.577	0.583	0.605	0.614
		2	0.701	0.704	0.709	0.730	0.735
		3	0.777	0.779	0.784	0.804	0.821
	1.2	0	0.190	0.191	0.197	0.205	0.212
		1	0.371	0.372	0.386	0.403	0.416
		2	0.488	0.490	0.505	0.523	0.537
		3	0.574	0.575	0.593	0.613	0.634
	1.1	0	0.049	0.050	0.056	0.063	0.065
		1	0.116	0.117	0.132	0.148	0.148
		2	0.171	0.173	0.194	0.218	0.224
		3	0.217	0.219	0.246	0.276	0.284
	0.9	0	0.060	0.068	0.070	0.074	0.041
		1	0.133	0.153	0.158	0.167	0.116
		2	0.190	0.218	0.226	0.237	0.184
		3	0.239	0.274	0.283	0.297	0.222
	0.8	0	0.240	0.243	0.244	0.248	0.228
		1	0.448	0.453	0.457	0.463	0.466
		2	0.577	0.583	0.588	0.595	0.583
		3	0.662	0.668	0.673	0.680	0.678
	0.7	0	0.440	0.446	0.447	0.453	0.467
		1	0.680	0.689	0.690	0.698	0.722
		2	0.804	0.812	0.814	0.822	0.842
		3	0.875	0.881	0.882	0.888	0.911

Figures 2 and 3 show $\hat{P}(|\hat{\tau} - \tau| \leq 1)$ over a wider range of shifts. For all estimators, accuracy increases as the shift moves away from $\delta = 1$, approaching 0.96–0.99 for $\delta = 2.0$ and 0.94–0.95 for $\delta = 0.5$. The curves of the four exact estimators nearly coincide; the largest visible gap occurs for $\delta = 2.0$, where the ± 10% oracle raises the probability from 0.961 to 0.993 (MLE). This gain arises because, for large shifts, the conventional estimator occasionally places the change one or two subgroups too early or too late when the last in-control observations happen to be large; restricting λ_a to a narrow band around the true value prevents this.

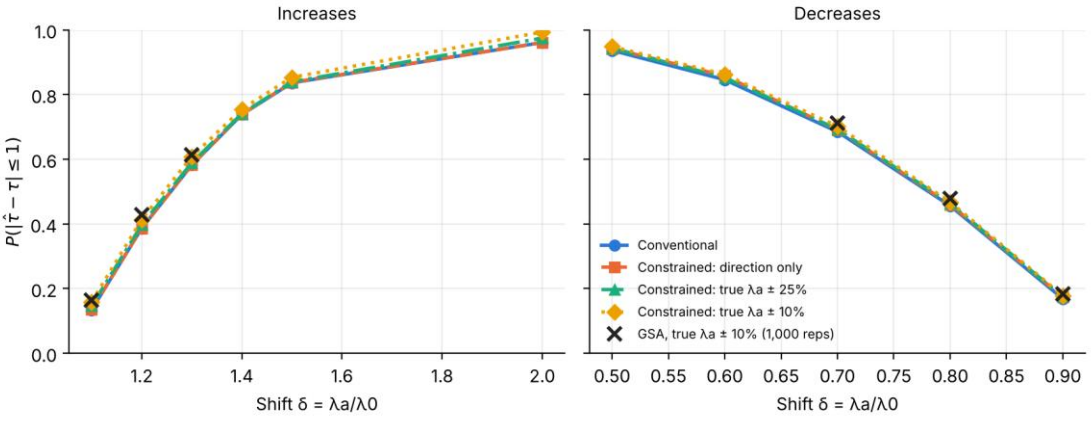


Fig. 2. Accuracy $\hat{P}(|\hat{\tau} - \tau| \leq 1)$ of the MLE-based estimators for increases (left) and decreases (right) in the rate parameter: conventional estimator, exact constrained estimators with three search intervals, and GSA (crosses, 1,000 replications).

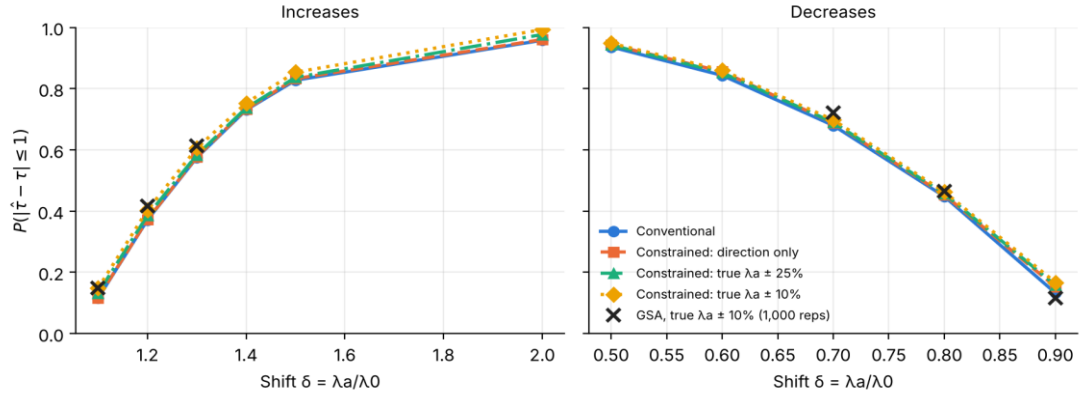


Fig. 3. Accuracy $\hat{P}(|\hat{\tau} - \tau| \leq 1)$ of the SIC-based estimators, as in Figure 2.

6. DISCUSSION

6.1. Effect of shift size

The behaviour of all estimators across shift sizes follows directly from the run-length properties of the C chart (Table 1). Large shifts ($\delta \geq 1.3$ or $\delta \leq 0.7$) are signalled within roughly 1 to 70 subgroups. The post-change observations then differ clearly from the in-control ones, and all estimators locate the change accurately. Small shifts ($\delta = 0.9$ and 1.1) produce long runs (ARL 650.4 and 94.6) in which in-control and out-of-control observations are hard to distinguish, so all estimators lose accuracy. The poorest results occur for $\delta = 0.9$, for which the chart is ARL-biased: a 10% decrease takes, on average, longer to detect than a false alarm. Accuracy for decreases improves sharply only when $\delta \leq 0.7$.

The large positive bias of the conventional estimators for $\delta = 1.1$ is consistent with the typical behaviour of likelihood-based change-point estimators for small shifts. When the shift is small, the first post-change observations are often indistinguishable from in-control data. The likelihood is then maximized at a later point that isolates a run of unusually large counts preceding the signal, with an estimate of λ_a well above its true value. The SIC estimator is affected more strongly than the MLE, because it must also estimate λ_0, and over a long series a segment boundary placed late is penalized little.

6.2. Source of the improvement

For a single step change, the conventional MLE (4) and SIC estimator (10) are obtained by evaluating the objective function at every candidate τ, with the rate parameters estimated in closed form. They are therefore the global optima of the unconstrained objective, and no search method, GSA included, can attain a higher likelihood or a lower SIC for that problem. Any gain of a metaheuristic estimator must therefore come from the way the problem is posed.

The present results make this explicit. When the rate parameter is restricted to an interval, the exact constrained estimator, computed by enumeration in milliseconds, improves on the conventional estimator, and the GSA solution of the same problem does not improve further: it finds the exact optimum in most replications, and where it does not, it is not systematically closer to the true change point. The interval acts as prior information. It excludes extreme estimates of λ_a that, in the unconstrained problem, allow the likelihood to be maximized at a misplaced τ, and the closer the interval is to the true value, the larger the benefit (Tables 3 and 4). In Bayesian terms, the constrained estimator is a maximum a posteriori estimator with a uniform prior on λ_a over the interval.

This interpretation has two practical implications. First, the benefit depends entirely on the quality of the prior information. An interval of ±10% around the *true* out-of-control rate is an oracle: if the engineer knew λ_a that precisely, the change point could be estimated with the MLE for known λ_a. With a realistic interval reflecting only the direction of the shift, the constrained estimators were practically indistinguishable from the conventional ones. Useful intervals may nevertheless be available in some applications, for example from process physics, from historical out-of-control episodes or from the shift size for which the chart was designed; in such cases the exact constrained estimator should be used. Second, for the single-change Poisson problem studied here, GSA offers no advantage over enumeration in accuracy, is non-deterministic, and requires about 15,000 fitness evaluations per replication instead of T. Its value lies in more complex problems — several change points, several unknown parameters, linear trends or multivariate processes — where enumeration grows combinatorially and a population-based search becomes a practical necessity. In such problems, the present results suggest that the search interval should be reported and justified as part of the estimator and that the exact solution of small instances should be used to verify that the metaheuristic reaches the global optimum.

6.3. Limitations and future research

The study has several limitations. (1) The constrained estimators depend on the search interval; an interval that does not contain the true value would bias the estimates, and only three intervals were studied here. A systematic study of interval width and of misspecified intervals is needed. (2) Only one in-control rate ($\lambda_0 = 20$), one change point ($\tau = 100$) and step changes were considered. Other values of λ_0, early and late change points, and linear-trend or multiple changes [7] should be investigated. (3) GSA was evaluated on 1,000 replications per scenario with one set of tuning parameters; other metaheuristics, such as genetic algorithms or particle swarm optimization, were not compared. (4) Results are limited to the C chart; estimators combined with CUSUM or EWMA charts [4, 5] may behave differently. (5) Confidence sets for the change point [3, 10] were not studied.

Future work could extend the constrained formulation to multiple change points, to unknown λ_0 in the MLE setting, to other count distributions such as the binomial and negative binomial, and to change points in multivariate and profile-monitoring problems, where enumeration is impractical and a population-based search offers a real computational advantage. Formulating the interval as an explicit prior distribution and comparing the resulting Bayesian estimators with the constrained estimators studied here would also be informative.

7. CONCLUSION

This paper examined the use of the gravitational search algorithm to optimize the likelihood function and the Schwarz information criterion for estimating the change point of a Poisson process after a C chart signal. A reproducible simulation study that included the exact constrained estimators as benchmarks showed that restricting the out-of-control rate to an interval improves the change-point estimates, mainly by reducing the bias for small shifts, and that the improvement is substantial only when the interval is narrow and centred close to the true rate. The improvement is due to this prior information and not to the search algorithm: GSA reproduced the exact

constrained optimum in most replications and did not improve on it. When only the direction of the shift is known, constrained and conventional estimators perform almost identically. For the single-change Poisson problem, enumeration with a constrained rate is therefore the method of choice; GSA and similar metaheuristics become valuable for multi-parameter and multiple-change problems, provided that their search intervals are explicitly justified.

Author's Contribution

The sole author was responsible for the conceptualization, methodology, software (Monte Carlo simulation, exact enumeration and gravitational search algorithm), formal analysis, and writing of the manuscript, and approved the final version.

Declaration of Generative AI Use

During the preparation of this work, the author used Claude only to improve the language and readability of the manuscript. The tool was not used to design the study, generate or analyze data, or produce results. After using it, the author reviewed and edited the content as needed and takes full responsibility for the content of the publication.

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Declaration of Competing Interests

The author declares no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Ethical Considerations

Not applicable. This study is based entirely on computer-simulated data and did not involve human participants, animals or personal data. Therefore, ethical approval and informed consent were not required.

Data and Code Availability

No empirical data were used. All results were generated by Monte Carlo simulation ($\lambda_0 = 20$, $\tau = 100$, $\delta = 0.5$–2.0, 10,000 replications per scenario) using fixed random seeds. The simulation code, including the exact enumeration estimator and the GSA implementation, is available from the author upon reasonable request

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